Topic 6 - Circular Motion and Gravitation

6.1 Circular Motion

Equations

- Linear speed - angular speed relationships:
  - $v = \omega r$
- Centripetal acceleration:
  - $a = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2}$
- Centripetal force:
  - $F = \frac{mv^2}{r} = m\omega^2 r$

Angular displacement

- In circular motion the velocity has a constant magnitude but a changing direction. This means that the vector is no longer constant, and therefore the object is accelerating.
- The angle moved around the circle by an object from where its circular motion starts is known as the angular displacement.
  - It's not a vector such as linear displacement.
  - measured in degrees (°) or radians (rad)

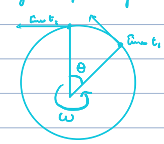
Angular speed

- Angular speed is the speed of an object in a circle. It has the sign "ω".
- The formula to find it is:
  - average angular speed = $\frac{\text{angular speed}}{\text{time for the angular displacement to happen}}$

$$\omega = \frac{\theta}{t} = \frac{\theta}{t_2 - t_1}$$

  - θ is the angular displacement, and t is the time taken for the angular displacement

Angular speed diagram



Period and frequency

- The time taken for an object to go around in a circle once is known as the periodic time / period of the motion (T).
- One period, the angular distance travelled is 2π radians:
  - $T = \frac{2\pi}{\omega}$ (rad s^{-1})
- Frequency is the number of times an object goes round a circle in unit time (usually 1 second).
  - $T = \frac{1}{f}$
  - $\omega = 2\pi f$

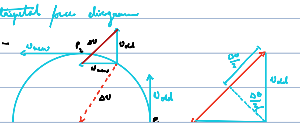
Worked Example

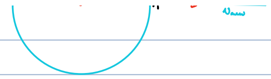
- 4.2 m long
  - $\omega = \frac{2\pi \text{ (full rotation)}}{3600 s \text{ (1 hour in seconds)}}$
    $= 0.00175$ rad s^{-1}
  - $\omega = \frac{\theta}{t}$
    $\theta = t\omega$
    $\theta = (20 \cdot 60) \cdot \left(\frac{2\pi}{3600}\right)$
    $\theta \approx 2.1$ rad
  - $\theta = t\omega$
    $= 150 \cdot \left(\frac{2\pi}{3600}\right)$
    $\theta \approx 15.71$ rad
  - $v = r\omega$
    $v = (1.4 m)\left(\frac{2\pi}{3600}\right)$
    $= 0.00733$ m s^{-1}
    $v = 7.33$ m s^{-1}

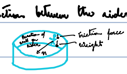
Linking angular and linear speeds

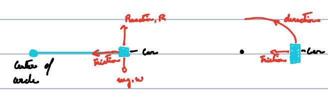
- When the circle has a radius "r" the circumference is $2\pi r$, and T, is the time taken to go around once. So the linear speed of the object along the edge of the circle is:
  - $v = \frac{2\pi r}{T} \longrightarrow T = \frac{2\pi r}{v} \longrightarrow \frac{2\pi r}{v} = \frac{2\pi}{\omega}$, $v = \omega r$
    $T = \frac{2\pi}{\omega}$
  - $s = \theta r$ (s = arc length of a circle)

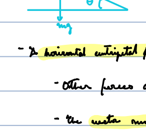
Centripetal acceleration

- Centripetal force diagram
  - 



- The diagram shows two points P_1 & P_2 on the circle together with the velocity vectors v_{old} & v_{new}.
  - The vectors are the same length as each other because the speed is constant.
  - However, v_{old} & v_{new} point in two different directions because the object moved round the circle by an angular distance $\Delta\theta$ between P_1 & P_2.
  - The Δv vector has to be added to v_{old} in order to make it become v_{new}.
- The time, Δt, to go between P_1 & P_2, and the linear distance around the circle between P_1 & P_2 are related by:
  - $\Delta t = \frac{r\Delta\theta}{v} \longrightarrow a = \frac{v^2}{r} \frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}$
- When $\Delta\theta$ is very small, the ratio $\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}$ is almost exactly equal to 1 and so the instantaneous acceleration a when P_1 & P_2 are very close together is:
  - $a = \frac{v^2}{r} = \omega^2 r = v\omega$ directed to the center of the circle.
  - The acceleration is at 90° to the velocity vector and it points inwards to the center of the circle.
- The force that acts to keep the object moving in a circle is called the centripetal force which leads to a centripetal acceleration.
- Centripetal force
  - The magnitude of the force (this is the force applied on the string (tension)) $= m\frac{v^2}{r} = m\omega^2 r = mv\omega$.
    - The direction of this force must be along the radial line between the object and the center of the circle.
  - This force is provided by another force (e.g friction).
  - Centripetal accelerations and forces in action
    - Satellites in orbit
      - Gravitational forces act between the center of mass of the Earth and the center of mass of the satellite. The direction of the force acting the satellite is always towards the center of the planet and the gravity supplies the centripetal force.
    - Amusement park rides
      - When the rotation speed is large enough the people are forced to the sides of the drum. They're "held" against the inside of the drum as the reaction at the wall provides the centripetal force to keep them moving in a circle.
        - Friction between the rider & the wall prevents the rider from slipping off the wall.
        - 

    - Turning and banking
      - A resultant force must act towards the center of the circle to provide a centripetal force for when a car turns a corner.
        - The car is in vertical equilibrium but not in horizontal equilibrium.
- Turning on a horizontal road
  - For a horizontal road surface, the friction acting between the tires and the road becomes the centripetal force.
  - The friction force is related to the coefficient of friction and the normal reaction at the surface where friction occurs.
  - 

  - If the car doesn't skid, the centripetal force required has to be less than frictional force
    - $m\frac{v^2}{r} < \mu_s mg$
      - μ_s is the static coefficient of friction.
  - When the car is already skidding the "<" becomes an equality and the dynamic coefficient of friction should be used.
    - Rearranged, this gives a max speed of $v_{max} = \sqrt{\mu_d rg}$ for circle of radius r.
- Banking
  - When the road is banked the normal reaction force contributes to the centripetal force that is needed for the vehicle to go round the track at a particular speed.
  - Diagram
    - 

    - A horizontal centripetal force directed towards the center of the circle is needed for the rotation.
      - Other forces are normal force to the surface (at angle θ) and its weight acting vertically down.
      - The vector sum of the horizontal components of the weight and the normal force must equal the centripetal force.
        - The horizontal components are the mg and normal force added together to give the centripetal force.

- Ask Ms. Jones to explain the sentence above. What are the horizontal components of the weight? (p. 254)
  - Another way to look at this is N is the normal force then the centripetal force is equal to: $N \sin\theta$.
- The normal force resolved vertically is $N\cos\theta$, and is equal & opposite to mg.
  - $F_{centripetal} = \frac{mv^2}{r} \longrightarrow \tan\theta = \frac{v^2}{gr}$
- <u>Moving in a vertical circle</u>
  - <u>What are the forces acting when the motion is in a vertical circle</u>
    - When a mass is attached to a string, the weight acts downwards when the mass is at the lowest point of the circle the tension will be pointing towards the center of the circle while the weight will be pointing down.
      - Formula $\longrightarrow T_{up} - mg = m\frac{v^2}{r} \longrightarrow T_{up} = m\frac{v^2}{r} + mg$
    - When the mass arrives at halfway between the lowest and highest part of the rotation, the tension will still be pointing towards the center and again the mass will be pointing down.
    - When the mass eventually reaches the top both the weight and the tension will be acting downwards:
      - Formula $\longrightarrow T_{down} + mg = m\frac{v^2}{r}$

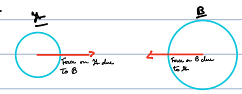
        $T_{down} = m\frac{v^2}{r} - mg$
      - Since the weight and the tension are in the same direction, they'll combine to provide the centripetal force, therefore, the tension required is less than tension T when the mass is at the lowest point or when its at the horizontal point.
    - As the mass moves along the circle, the tension will vary. The lowest tension will occur at the top of the circle and max at the bottom.
      - The bottom is the point where the string is most likely to break. This will occur if the max breaking tension of the string is T_{break}.
      - For the string not to break, the tension in the string must be:
        - $T_{break} > m\frac{v^2}{r} + mg$
      - and the linear speed at the bottom of the circle must be less than:
        - $v = \sqrt{\frac{r}{m}(T_{break} - mg)}$
    - (p. 256) How are we supposed to prove that the wheels will lose contact with the bridge when the speed is equal to $\sqrt{ga}$?
  - <u>How does speed change when motion is in a vertical circle?</u>
    - As the mass in the string moves to the top of the circle it will start slowing down. This is because of the fact that the kinetic energy of the mass on the string will be converted to gravitational potential energy. This same behavior is seen with pendulums in SHM as well as other circular motions in a vertical circle.
      - If the mass at the top will slow down by too much, then the mass will slow down and eventually fall down.
      - To maintain the centripetal force (F_c) the motion needed is: $F_c = m\frac{v^2}{r}$. If F_c is supplied entirely by gravity then $F_c = mg = m\frac{v^2}{r}$ meaning for an instant the object will be in free-fall, rearranging the equation to $v_{top} = \sqrt{gr}$.
        - This is the minimum speed at the top of the circle where the motion is still circular. The min speed doesn't depend on mass.
    - Since energy is conserved, kinetic energy at top + Δ gravitational potential energy between top & bottom = kinetic energy at bottom. $\longrightarrow \frac{1}{2}mv^2 + mg\,2r = \frac{1}{2}mv^2_{bottom} \longrightarrow T_{bottom} = T_{top} + 6m$
  - <u>Worked example</u>
    - 7.5 times in 5.2 seconds, $r = 4.2\,m$, $m = 4\,kg$
      - Average angular speed (ω).
      - $\omega = \frac{\theta}{t}$

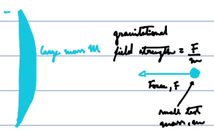
        $= \frac{2\pi \cdot 7.5}{5.2}$

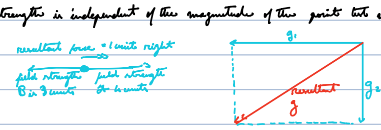
        $= 9.06\ rad\ s^{-1}$
      - $T = m\omega^2 r$

        $= (4)(9.06)^2(4.2)$

        $T = 1400$ Newtons
    - An assumption that has to be made is that wind resistance is zero, meaning that no energy will be lost from the system. Furthermore, another assumption that we are making is that the force applied on the object is constant throughout while in reality its variable. Finally, another assumption is that the circle is in fact horizontal while in most cases its "un't" to achieve max distance.
- <u>6.2 Newton's law of gravitation</u>
  - <u>Equations</u>
    - <u>Newton's law of gravitation:</u>
      - $F = G\frac{Mm}{r^2}$
    - <u>Gravitational field strength:</u>
      - $g = \frac{F}{m}$
    - <u>Gravitational field strength and the gravitational constant:</u>
      - $g = G\frac{M}{r^2}$
  - <u>Gravitational field strength</u>
    - Gravity acts at a distance and is an example of a force that has an associated force field.

- E.g. 2 masses, mass 1 (A) & mass 2 (B). Mass A is in the force field due to the second mass (B) and a force acts on A. B is in gravitational field of A and also experiences a force.
  - These two forces are equal to one another, they have the same magnitude (even if masses are different) but act in opposite directions.
  - 

- If the sizes of the two objects are very small e.g. size of humans, then the force of gravity will be extremely weak. Only when one of the masses is as large as a planet does the force become noticible.
  - However, no matter the size of the mass it will still apply a gravitational field on every other mass in the universe.
- Gravitational forces are the weakest of the fundamental forces, Therefore, they require large amounts of mass for the force to felt.
  - Strength of gravitational field is defined using the idea of a small test mass. The test mass must be so small that it doesn't produce its own gravitational field. If mass is large, then it'll exert a force of its own on the mass that produces the field being measured. The mass will accelerate the other mass & alter the arrangement that is being measured.
- 

- Defining gravitational field strength
  - If the mass of the test mass is m, and the field is producing a gravitational force of F on the test mass, then the gravitational field strength (g) is: $g = \frac{F}{m}$.
    - Units = $N kg^{-1}$.
  - Gravitational field strength at a point is the force per unit mass experienced by a small point mass placed at that point.
  - Field strength is independent of the magnitude of the point test mass. so vector field strength can be added together.
    - 

- g and the acceleration due to gravity
  - "g" is used for the gravitational field strength as well as acceleration due to gravity.
    - Gravitational field strength is: $g = \frac{F}{m}$ and acceleration due to gravity is: $a = \frac{F}{m}$ $(F = ma)$ $\longrightarrow g = \frac{F}{m}$ showing that acceleration due to gravity = gravitational field strength (g).
      - $N kg^{-1} \equiv m s^{-2}$ ("≡" means equivalent to).

- Newton's law of gravitation - an inverse - square law
  - The gravitational force F between two objects with masses M and m whose centres are separated by distance are:
    - always attractive to one another
    - proportional to $\frac{1}{r^2}$
      - As the distance increases the force between the masses decreases by a square law. E.g: distance increases by two, force decreases to $\frac{1}{4}$ of its original value.
    - proportional to M & m
  - Summed up in equation: $F \propto \frac{Mm}{r^2}$
  - "G" is the universal gravitational constant which is a numerical constant of proportionality.
    - Its formula is: $F = \frac{GMm}{r^2}$
    - Its value is $6.67 \cdot 10^{-11} N m^2 kg^{-2}$.
    - Gravity is always attractive so if the distance is measured from the center of mass M to mass m then the force on m due to M is towards M.
      - In other words, the force is in the opposite direction to the direction in which the distance is measured.

- Worked Example
  - 100g apple, Earth = $6 \cdot 10^{24} kg$, r = 6.4 mm
    - $F = \frac{GMm}{r^2}$

      $$= \frac{(6.67 \cdot 10^{-11})(6 \cdot 10^{24})(0.1)}{(6.4 \cdot 10^{6})^2} \quad \frac{N m^2 kg^{-2} \cdot kg \cdot kg}{m^2} = \text{Newtons}$$

      F = 0.977050 7813

      F = 1.0 Newton
  - Proton m = $1.7 \cdot 10^{-27} kg$, electron m = $9.1 \cdot 10^{-31} kg$, d = $1.5 \cdot 10^{-10}$
    - $F = \frac{GMm}{r^2}$

      $$F = \frac{(6.67 \cdot 10^{-11})(1.7 \cdot 10^{-27})(9.1 \cdot 10^{-31})}{(1.5 \cdot 10^{-10})^2}$$

      $F = 4.59 \cdot 10^{-48} N$
  - The reason that the radius is used for the first problem is because the apple is $6.4 \cdot 10^{6}$ m away from the center of the Earth (where the measurement is made from). The reason that the

$r = 1.5 \cdot 10^{-10}$ m is because r isn't the radius but the distance between the two objects.

- Gravitational field strength re-visited
  - The field strength at a distance r from a point mass, M
    - Single point mass M placed a long way from any other mass.
    - The magnitude of the force F between the two masses M and m is
      - $F = \frac{GMm}{r^2}$
    - So the gravitational field strength g is
      - $g = \frac{F}{m} = \frac{GM}{r^2}$
  - The field strength at a distance r from the centre outside a sphere of mass M
    - g outside a spherical planet is: $g = \frac{GM}{r^2}$.
      - If we're outside the sphere, all the mass acts as though it is a point mass of size M positioned at the centre of mass.
  - Linking orbits and gravity
    - The gravitational force of a planet provides the centripetal force to keep a satellite in orbit.
    - Worked Example
      - $g = \frac{GM}{r^2}$
        $= \frac{(6.67 \cdot 10^{-11})(7.3 \cdot 10^{22})}{(1.7 \cdot 10^{6})^2}$
        $g = 1.69 \text{ N kg}^{-1}$
      - $g = \frac{GM}{r^2}$
        $= \frac{(6.67 \cdot 10^{-11})(2 \cdot 10^{30})}{(1.5 \cdot 10^{11})^2}$
        $g = 0.00593 \text{ N kg}^{-1}$
        $g = 6.0 \text{ mN kg}^{-1}$
    - Newton said that if an object had enough of a velocity, it would match the curvature of the Earth with its own curvature, keeping it in orbit.
      - The gravitational attraction F_g provides the centripetal force F_c:
        - $F_c = F_g = \frac{mv^2}{r} = \frac{GM_E m}{r^2}$
          - M_E is mass of Earth, r is distance from the satellite to the centre of the Earth, v is the linear speed of the satellite, and m is its mass.
        - $v = \sqrt{\frac{GM_E}{r}}$ (speed of satellite at particular radius).
        - $\omega^2 = \frac{GM_E}{r^3}$ (ω is the angular speed)
        - $T^2 = \frac{4\pi^2 r^3}{GM_E}$ (T is the orbital period)

          - (orbital period of a satellite)^2 $\propto$ (orbital radius)^3 known as Kepler's Third Law
    - Worked example
      - $M_{sun} = 2 \cdot 10^{30}$ kg, $r = 7.8 \cdot 10^{11}$ m
        - $T^2 = \frac{4\pi^2 r^3}{GM_E}$
          $= \frac{4\pi^2 (7.8 \cdot 10^{11})^3}{(6.67 \cdot 10^{-11})(2 \cdot 10^{30})}$
          $T = 374751912.2$
          $T = 3.7 \cdot 10^{8}$ s

        - $T = 3.2 \cdot 10^{7}$ s (Earth orbital period around sun)
          - $T^2 = \frac{4\pi^2 r^3}{GM_E}$
            $(3.2 \cdot 10^{7})^2 = \frac{4\pi^2 (1.5 \cdot 10^{11})^3}{(6.67 \cdot 10^{-11}) M_E}$
            $M_E = \frac{4\pi^2 (1.5 \cdot 10^{11})^3}{(6.67 \cdot 10^{-11})(3.2 \cdot 10^{7})^2}$
            $M_E = 1.95 \cdot 10^{30}$ kg

            $T^2 = \frac{4\pi^2 r^3}{GM_E}$
            $T^2 = \frac{4\pi^2 (2.3 \cdot 10^{11})^3}{(6.67 \cdot 10^{-11})(1.95 \cdot 10^{30})}$
            $T = 6.1 \cdot 10^{7}$ s
          - Another solution: $\frac{T_m^2}{T_E^2} = \frac{r_m^3}{r_E^3}$ How are you supposed to solve this problem with this equation? (p. 264)